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UPPER BOUNDS FOR EIGEN VALUES OF LAPLACE OPERATOR ON MANIFOLD

Manju Yadav
Lecturer, Jayoti Vidyapeeth Women's University, Jaipur
Mnjyadav89@gmail.com

Abstract - In this paper, we are concerned with upper bounds of eigen-values on manifolds. Eigen-values have many applications in geometry and in other fields of mathematics. We develop a universal approach to upper bounds on both continuous and discrete structures based upon certain properties of the corresponding heat kernel. we start with a well-defined Laplace operator Δ on functions on M so that Δ is a self-adjoint operator in $L^2(M, +)$ with a discrete spectrum and a distance function dist(x, y) on M.
Keywords: eigen-values, Laplace transform, heat equation, manifold

1 INTRODUCTION

In this paper, let us consider Laplace operator on smooth compact Riemannian manifold M, with metric g. since M has boundary ∂M, then we require in addition that g vanishes at the boundary. This defines the Laplacian with drichilet boundary condition .the Laplace operator is a self-adjoint operator, so by spectrum theorem there is a sequence of eigen-values

$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$

And an orthogonal basis $\phi_1, \phi_2, \ldots$ of $L^2(M)$, which are eigenfuntions of Laplace operator.

Laplacian Operator On Riemannian Manifold:

The laplacian operator on a Riemannian manifold (M , g) is a function defined as

$\Delta_g : C^\infty(M) \rightarrow C^\infty(M)$

defined as $\Delta_g = -\mathrm{div}_g . \nabla_g$

Since both ∇_g and div_g are linear operators it follows that for any $\phi, \psi \in C^\infty(M)$

$\Delta_g(\phi + \psi) = \Delta_g\phi + \Delta_g\psi$.

in addition we have

$$\Delta_g(\phi.\psi) = \psi\Delta_g\phi + \phi\Delta_{g\psi} - 2\langle\Delta_g\phi, \Delta_g\psi\rangle$$

Eigen Values of Laplace Operator On Manifold:

Let M be a smooth connected compact Riemannian manifold and Δ be a Laplace operator associated with the Riemannian metric i.e. in coordinates $x_1, x_2, \ldots, x_n$

$$\Delta u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{g}\, g^{ij} \frac{\partial u}{\partial x_j}\right)$$

Where g^{ij} are contra-variant components of the metric tensor and $g = \det\|g_{ij}\|$ and u is a smooth function on M.

Theorem: Suppose that we have chosen k+1 disjoint subsets $X_1, X_2, \ldots, X_{K+1}$ of M such that the distance between any pair of them is at least D > 0. Then for any k > 1

$$\lambda_k - \lambda_0 \leq \frac{1}{D^2} \max_{i \neq j} \left(\log \frac{4}{\int_{X_i} \phi_0^2 \int_{X_j} \phi_0^2}\right)^2$$

Proof: The proof is based upon two fundamental facts about the heat kernel p(x, y, t) being by definition the unique fundamental solution to heat equation

$$\frac{\partial u}{\partial t} u(x,t) - \Delta u(x,t) = 0 \qquad (1)$$

With the boundary condition

$$\alpha u + \beta \frac{\partial u}{\partial v} = 0$$

P(x, y, t) can be written in the form

$$p(x,y,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i(x) \phi_i(y) \qquad (2)$$

For any two disjoint Boral sets X, Y⊂ M where D = dist(X, Y).

First we take the particular case k = 2. We start with integrating the eigenvalue expansion (2) as follows

$$I(f,g) \equiv \int_X \int_Y P(x,y,t) f(x) g(y) \mu(dx) \mu(dy) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \int_X f\phi_i \int_Y g\phi_i \qquad (4)$$

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Let us denote by f_i the Fourier coefficients of the function $f1_x$ with respect to the frame $\{\phi_i\}$ and by g_i the Fourier coefficients of the function $g1_y$ .Then

$$I(f,g)=e^{-\lambda_0 t}f_0 g_0+\sum_{i=1}^{\infty}e^{-\lambda_i t}f_i g_i \ge e^{-\lambda_0 t}f_0 g_0 - e^{-\lambda_1 t}\|f1_X\|_2\|g1_Y\|_2 \quad (5)$$

Where we used the fact that

$$\left|\sum_{i=1}^{\infty}e^{-\lambda_i t}f_i g_i\right| \le e^{-\lambda_1 t}(\sum_{i=1}^{\infty}f_i^2\sum_{i=1}^{\infty}g_i^2)^{1/2}$$

since

$$e^{-\lambda_1 t}(\sum_{i=1}^{\infty}f_i^2\sum_{i=1}^{\infty}g_i^2)^{1/2} \le e^{-\lambda_1 t}\|f1_X\|_2\|g1_Y\|_2$$

Putting into (3)-

$$I(f,g)\le\|f1_X\|_2\|g1_Y\|_2\exp(-\frac{D^2}{4t}-\lambda_0 t)$$

From (5)-

$$I(f,g)\ge e^{-\lambda_0 t}f_0 g_0 - e^{-\lambda_1 t}\|f1_X\|_2\|g1_Y\|_2$$

$$e^{-\lambda_0 t}f_0 g_0 - e^{-\lambda_1 t}\|f1_X\|_2\|g1_Y\|_2 \le \|f1_X\|_2\|g1_Y\|_2\exp(-\frac{D^2}{4t}-\lambda_0 t)$$

$$-e^{-\lambda_1 t}\|f1_X\|_2\|g1_Y\|_2 \le -e^{-\lambda_0 t}f_0 g_0 + \|f1_X\|_2\|g1_Y\|_2\exp(-\frac{D^2}{4t})e^{-\lambda_0 t}$$

$$e^{-(\lambda_1-\lambda_0)t}\|f1_X\|_2\|g1_Y\|_2 \ge f_0 g_0 - \|f1_X\|_2\|g1_Y\|_2\exp(-\frac{D^2}{4t}) \quad (7)$$

Let us choose

$$t=\frac{D^2}{4\log\frac{2\|f1_X\|_2\|g1_Y\|_2}{f_0 g_0}}$$

Putting into (7) we get:

$$e^{-(\lambda_1-\lambda_0)t}\|f1_X\|_2\|g1_Y\|_2 \ge \frac{1}{2}f_0 g_0$$

$$-(\lambda_1-\lambda_0)t \ge \log\frac{f_0 g_0}{2\|f1_X\|_2\|g1_Y\|_2}$$

$$\lambda_1-\lambda_0 \le \frac{1}{t}\log\frac{2\|f1_X\|_2\|g1_Y\|_2}{f_0 g_0}$$

Putting the value of t,

$$\lambda_1-\lambda_0 \le \frac{4}{D^2}(\log\frac{2\|f1_X\|_2\|g1_Y\|_2}{f_0 g_0})^2 \quad (8)$$

Finally, we choose $f = g = \phi_0$ such that

$$f_0=\int_X f\phi_0=\int_X \phi_0^2$$

And

$$\|f1_X\|_2=(\int_X \phi_0^2)^{1/2}=\sqrt{f_0}$$

Similarly

$$g_0=\int_Y g\phi_0=\int_Y \phi_0^2$$

$$\|g1_Y\|_2=(\int_Y \phi_0^2)^{1/2}=\sqrt{g_0}$$

Putting into (8)

$$\lambda_1-\lambda_0 \le \frac{4}{D^2}[\log(\frac{2\sqrt{f_0}\sqrt{g_0}}{f_0 g_0})]^2$$

This implies:

$$\lambda_1-\lambda_0 \le \frac{1}{D^2}(\log\frac{4}{\int_X \phi_0^2\int_Y \phi_0^2})^2 \quad (9)$$

Now we turn to the general case k > 2 let us consider a function f(x) and denote by f_i^j the i^{th} Fourier coefficient of the function $f1_x$ i.e.,

$$f_i^j=\int_{X_J} f\phi_i$$

$$I_{lm}(f,f)=\int_{X_l}\int_{X_m}p(x,y,t)f(x)f(y)\mu(dx)\mu(dy)$$

Then we have the upper bound for $I_{lm}(f, f)$

$$I_{lm}(f,f)\le\|f1_{X_l}\|_2\|f1_{X_m}\|_2\exp(-\frac{D^2}{4t}-\lambda_0 t) \quad (10)$$

While we rewrite the lower bound (5) in another way:

$$I_{lm}(f,f)\ge e^{-\lambda_0 t}f_0^l f_0^m+\sum_{i=1}^{k-1}e^{-\lambda_0 t}f_i^l f_i^m - e^{-\lambda_k t}\|f1_{X_l}\|_2\|f1_{X_m}\|_2 \quad (11)$$

Now we want to kill the middle term on the right-hand side (11) by choosing appropriate l and m.

Let us consider k+1 vectors $f^m=(f_1^m, f_2^m, \ldots, f_{k-1}^m)$ M = 1, 2, …, k+1 in R^{k-1} and let us supply this (k-1)-dimensional space with a scalar product given by

$$(v,w)=\sum_{i=0}^{k-2}v_i w_i e^{-\lambda_{i+1}t}$$

Let us apply the following elementary fact: out of any k+1 vector in (k-1) dimensional Euclidean space there are always two vectors with non-negative scalar product. Therefore, we can find different l and m so that $(f^l, f^m) \ge 0$ and due to this choice we

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are able to cancel the second term on the right hand side (11).

Comparing (10) and (11) we get

$$e^{-(\lambda_k-\lambda_0)t}\|f1_{X_l}\|_2\|f1_{X_m}\|_2 \le f_0{}^l f_0{}^m - \|f1_{X_l}\|_2\|f1_{X_m}\|_2 \exp(-\frac{D^2}{4t}) \quad (12)$$

Now similar to the case k = 2 we choose t such that

$$t = \min_{l\neq m} \frac{D^2}{4\log \frac{2\|f1_{X_l}\|_2 \|f1_{X_m}\|_2}{f_0{}^l f_0{}^m}}$$

Putting the value of t into (12) we get,

$$e^{-(\lambda_k-\lambda_0)t}\|f1_{X_l}\|_2\|f1_{X_m}\|_2 \ge \frac{1}{2} f_0{}^l f_0{}^m$$

Therefore,

$$\lambda_k - \lambda_0 \le \frac{1}{t}\log\frac{2\|f1_{X_l}\|_2 \|f1_{X_m}\|_2}{f_0{}^l f_0{}^m}$$

Putting the value of t,

$$\lambda_k - \lambda_0 \le \frac{4}{D^2}\max_{l\neq m}(\log\frac{2\|f1_{X_l}\|_2 \|f1_{X_m}\|_2}{f_0{}^l f_0{}^m})^2 \quad (13)$$

Now we taking f = φ_0 such that,

$$f_0{}^l = \int_{X_l} f\phi_0 = \int_{X_l} \phi_0{}^2$$

And

$$\|f1_{X_l}\|_2 = (\int_{X_l}\phi_0{}^2)^{1/2} = \sqrt{f_0{}^l}$$

Similarly

$$f_0{}^m = \int_{X_m} f\phi_0 = \int_{X_m} \phi_0{}^2$$

$$\|f1_{X_m}\|_2 = (\int_{X_m}\phi_0{}^2)^{1/2} = \sqrt{f_0{}^m}$$

Putting into (13) we get,
Thus for any two disjoint subset of M, we have

$$\lambda_k - \lambda_0 \le \frac{1}{D^2}\max_{l\neq m}(\log\frac{4}{\int_{X_l}\phi_0{}^2\int_{X_m}\phi_0{}^2})^2$$

What was to be proved.

$$\lambda_k - \lambda_0 \le \frac{1}{D^2}\max_{i\neq j}(\log\frac{4}{\int_{Xi}\phi_0{}^2\int_{X_j}\phi_0{}^2})^2$$

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